Student Workbook

Continuity

Name: ________________________________ Date: _________________ Class: _________

ESSENTIAL QUESTION

How do we verify a function is continuous, classify the types of discontinuity, and use continuity to guarantee that roots and solutions exist?

Lesson Overview

A function is continuous at a point $x = c$ when its graph has no holes, jumps, or vertical asymptotes at that point — you can draw through it without lifting your pencil. Formally, three conditions must all hold simultaneously: $f(c)$ is defined, $\lim_{x\to c} f(x)$ exists, and $\lim_{x\to c} f(x) = f(c)$. If any one condition fails, the function is discontinuous at $x = c$.

$$f(c) \text{ defined}$$
$$\lim_{x\to c} f(x) \text{ exists}$$
$$\lim_{x\to c} f(x) = f(c)$$

Removable	The limit exists but $f(c)$ is undefined or $f(c) \neq \lim f(x)$ — a 'hole' in the graph.
Jump	The left- and right-hand limits both exist but are not equal.
Infinite	$f(x)$ approaches $\pm\infty$ as $x\to c$, producing a vertical asymptote.
IVT	If f is continuous on $[a,b]$ and N is between $f(a)$ and $f(b)$, some c in (a,b) has $f(c) = N$.

Worked Examples

EXAMPLE 1

Verify that $f(x) = x^2 - 1$ is continuous at $x = 2$ using the three-condition test.

- Condition 1 — $f(2) = (2)^2 - 1 = 3$. Defined. ✓
- Condition 2 — the function is a polynomial, so $\lim_{x\to 2}(x^2-1) = 3$. ✓
- Condition 3 — $\lim_{x\to 2} f(x) = f(2)$: $3 = 3$. ✓

Answer: f is continuous at $x = 2$.

EXAMPLE 2

Identify and classify the discontinuity of $f(x) = (x^2 - 4)/(x - 2)$ at $x = 2$.

- Condition 1 — denominator = 0, so $f(2)$ is undefined. ✗
- Condition 2 — factor: $(x+2)(x-2)/(x-2) = x+2$ for $x\neq 2$, so $\lim_{x\to 2} f(x) = 4$. Limit exists.
- The limit exists but $f(2)$ is undefined → removable discontinuity.

Answer: Removable discontinuity at $x = 2$ (hole at the point $(2, 4)$).

EXAMPLE 3

Determine if $f(x) = \{x+1 \text{ if } x<3;\ 2x-2 \text{ if } x\geq 3\}$ is continuous at $x = 3$.

- Condition 1 — $f(3) = 2(3)-2 = 4$. Defined. ✓
- Condition 2 — left limit: $3+1=4$. Right limit: $2(3)-2=4$. Both equal 4. ✓
- Condition 3 — $4 = 4$. ✓

Answer: f is continuous at $x = 3$.

EXAMPLE 4

Find the value of k that makes $f(x) = \{kx+1 \text{ if } x\leq 2;\ x^2-1 \text{ if } x>2\}$ continuous at $x = 2$.

- Left-hand limit: $\lim(x\to 2^-)(kx+1) = 2k+1$. Right-hand limit: $\lim(x\to 2^+)(x^2-1) = 3$.
- $f(2) = 2k+1$ (using $x\leq 2$ piece). Set left limit = right limit: $2k+1 = 3 \to 2k = 2$.

Answer: $k = 1$

EXAMPLE 5

Use the IVT to show that $f(x) = x^3 - x - 1$ has a root on [1, 2].

- f is a polynomial, so it is continuous on [1,2]. ✓
- $f(1) = 1-1-1 = -1$ (negative). $f(2) = 8-2-1 = 5$ (positive).
- Since $f(1) < 0 < f(2)$, by the IVT there exists $c \in (1,2)$ with $f(c) = 0$.

Answer: By the IVT, f has at least one root in (1, 2).

Guided Practice

Work through each problem below. Use the hint if you get stuck — write your final answer on the last line.

1 Check the continuity of $f(x) = (x + 3)/(x - 1)$ at $x = 1$ and at $x = 3$.

Hint: At $x = 1$, check whether the denominator is zero. At $x = 3$, evaluate all three conditions.

2 Classify the discontinuity of $f(x) = |x|/x$ at $x = 0$.

Hint: Compute the left-hand limit ($x\to 0^-$) and the right-hand limit ($x\to 0^+$) separately. Do they agree?

3 Determine whether $f(x) = \{2x-1 \text{ if } x\leq 1;\ x^2 \text{ if } x>1\}$ is continuous at $x = 1$.

Hint: Compute $f(1)$, $\lim(x\to 1^-)$, and $\lim(x\to 1^+)$. Check all three conditions.

4 Find k so that $f(x) = \{3x-k \text{ if } x<2;\ kx^2+1 \text{ if } x\geq 2\}$ is continuous at $x = 2$.

Hint: Set the left-hand limit equal to $f(2)$ and solve for k.

5 Use the IVT to confirm that $g(x) = x^4 - 3x + 1$ has a root on [0, 1].

Hint: Evaluate g(0) and g(1). Are they opposite in sign? Is g continuous on [0,1]?

Key Vocabulary

Continuity at a point	f(c) is defined, lim(x→c) f(x) exists, and lim(x→c) f(x) = f(c).
Removable Discontinuity	The limit exists but f(c) is undefined or ≠ lim f(x) — a 'hole' in the graph.
Jump Discontinuity	The left- and right-hand limits both exist but are not equal.
Infinite Discontinuity	f(x) approaches ±∞ as x→c, producing a vertical asymptote.
Intermediate Value Theorem (IVT)	If f is continuous on [a,b] and N is between f(a) and f(b), some c in (a,b) has f(c) = N.

Quick Check Quiz

Circle the letter of the correct answer for each question.

1 **Which of the following is NOT one of the three conditions required for f to be continuous at x = c?** Multiple Choice

- **A** f(c) is defined
- **B** lim(x→c) f(x) exists
- **C** f is differentiable at x = c
- **D** lim(x→c) f(x) = f(c)

2 **$f(x) = (x^2 - 9)/(x - 3)$ has what type of discontinuity at x = 3?** Multiple Choice

- **A** Jump discontinuity
- **B** Infinite discontinuity
- **C** Removable discontinuity
- **D** No discontinuity

3 **For $f(x) = \{x^2$ if $x \le 1$; $2x-1$ if $x>1\}$, what is true at x = 1?** Multiple Choice

- **A** Jump discontinuity because the pieces use different formulas
- **B** Continuous because f(1)=1, $\lim(x\to1^-)=1$, and $\lim(x\to1^+)=1$
- **C** Removable discontinuity because f(1) is defined by only one piece
- **D** Infinite discontinuity because the derivative is undefined

4 **The Intermediate Value Theorem guarantees a root of f on (a, b) when:** Multiple Choice

- **A** f is differentiable on [a,b] and $f(a) \neq f(b)$
- **B** f is continuous on [a,b] and $f(a) \cdot f(b) < 0$
- **C** f is increasing on [a,b]
- **D** f(a) and f(b) are both positive

5 **A function has a jump discontinuity at x = c. Which statement is true?** Multiple Choice

- **A** f(c) is undefined
- **B** The two-sided limit exists but does not equal f(c)
- **C** The left-hand and right-hand limits exist but are not equal
- **D** The function approaches infinity near x = c

Common Mistakes to Avoid

✗ Assuming a function is continuous just because the limit exists.	✓ The limit must exist AND equal f(c), and f(c) must be defined. All three conditions are required.
✗ Confusing removable and jump discontinuities — both have a 'break' in the graph.	✓ Removable: the limit exists (both sides agree) but ≠ f(c) or f(c) undefined. Jump: left and right limits exist but differ.
✗ Applying the IVT without verifying continuity on the interval first.	✓ The IVT requires f to be continuous on [a,b]. Always state this as the first step.
✗ Thinking the IVT tells you exactly where the root is.	✓ The IVT only guarantees existence of a root — it does not find the root's location.

Math Tips

★ Polynomials and rational functions are continuous everywhere they are defined. Rational functions are discontinuous only where the denominator equals zero.

★ To find a removable discontinuity, factor and cancel. The cancelled factor reveals the x-value of the hole.

★ For piecewise functions, always check continuity at the boundary points — that is where discontinuities can occur.

★ The IVT is an existence theorem: if you start below zero and end above zero on a continuous path, you must cross zero somewhere.